Problem j13-I-1/j13-I-15. *Denote by $d(n)$ the number of all natural divisors of a natural number n. Prove that the sequence $d(n^2+1)$ starting from any natural number n_0 is not strongly monotonic.* (Vilnius University)

Solution. Note that $d(n^2+1) < n$ for all even n. Indeed, the number n^2+1 is not square and so it is possible to split the set of all its divisors into pairs $\{d, (n^2+1)/d\}$ where $d < n$ and d is odd. The number of divisors in all such pairs does not exceed n.

Let us assume that starting from some $n_0 \in \mathbb{N}$, the sequence is strictly monotone. For $d(n^2+1)$ is always even, we get

$$d\big((n+1)^2+1\big) \geq d(n^2+1)+2$$

or, in general,

$$d\big((n+k)^2+1\big) \geq d(n^2+1)+2k$$

for any natural numbers $n \geq n_0$ and $k \geq 1$. Let $N \geq n_0$ (e.g., $N = n_0$). Taking any $s \geq N - d(N^2+1)$ (such that $N+s$ is even), we get

$$d\big((N+s)^2+1\big) \geq d(N^2+1)+2s \geq N+s\,,$$

which is a contradiction with $d\big((N+s)^2+1\big) < N+s$. □

Problem j13-I-2/j13-I-19. *Let $A = [a_{i,j}]$ be an $m \times n$ real matrix with at least one non-zero entry. For each $i \in \{1, \ldots, m\}$, let $R_i := \sum_{j=1}^{n} a_{i,j}$ denote the sum of the entries of the i-th row of A, and for each $j \in \{1, \ldots, n\}$, let $C_j := \sum_{i=1}^{m} a_{i,j}$ denote the sum of the entries of the j-th column of A. Prove that there exist indices $i_0 \in \{1, \ldots, m\}$ and $j_0 \in \{1, \ldots, n\}$ such that*

$$a_{i_0,j_0} > 0\,, \qquad R_{i_0} \geq 0\,, \qquad C_{j_0} \geq 0\,,$$

or

$$a_{i_0,j_0} < 0\,, \qquad R_{i_0} \leq 0\,, \qquad C_{j_0} \leq 0\,.$$

(Assistant (and former student competitor) Vjekoslav Kovač / University of Zagreb)

Solution. Consider the following sets of indices (some of which may be empty):

$$\begin{aligned} I^+ &:= \big\{\, i \in \{1, \ldots, m\} \;\big|\; R_i \geq 0 \,\big\}\,, \\ I^- &:= \big\{\, i \in \{1, \ldots, m\} \;\big|\; R_i < 0 \,\big\}\,, \\ J^+ &:= \big\{\, j \in \{1, \ldots, n\} \;\big|\; C_j > 0 \,\big\}\,, \\ J^- &:= \big\{\, j \in \{1, \ldots, n\} \;\big|\; C_j \leq 0 \,\big\}\,. \end{aligned}$$

Suppose that the statement of the problem does not hold. Then (but not equivalently) we have $a_{i,j} \leq 0$ for every $(i,j) \in I^+ \times J^+$ and we have $a_{i,j} \geq 0$ for every $(i,j) \in I^- \times J^-$. Let us write the sum $\sum_{(i,j)\in I^- \times J^+} a_{i,j}$ in two different ways:

$$\sum_{(i,j)\in I^- \times J^+} a_{i,j} = \sum_{i \in I^-} \Big(\sum_{j=1}^{n} a_{i,j} - \sum_{j \in J^-} a_{i,j} \Big) = \sum_{i \in I^-} R_i - \sum_{(i,j)\in I^- \times J^-} a_{i,j} \leq 0\,,$$

$$\sum_{(i,j)\in I^- \times J^+} a_{i,j} = \sum_{j \in J^+} \Big(\sum_{i=1}^{m} a_{i,j} - \sum_{i \in I^+} a_{i,j} \Big) = \sum_{j \in J^+} C_j - \sum_{(i,j)\in I^+ \times J^+} a_{i,j} \geq 0\,.$$

Therefore, $\sum_{(i,j)\in I^- \times J^+} a_{i,j} = 0$ and we have only equalities in the two formulae above. This is only possible if $\sum_{i\in I^-} R_i = 0$ and $\sum_{j \in J^+} C_j = 0$, so $I^- = \emptyset$ and $J^+ = \emptyset$,† which means $R_i \geq 0$ for all $i = 1, \ldots, m$ and $C_j \leq 0$ for all $j = 1, \ldots, n$. Moreover, from

$$0 \leq \sum_{i=1}^{m} R_i = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{i,j} = \sum_{j=1}^{n} C_j \leq 0\,,$$

we conclude $R_i = 0$ for $i = 1, \ldots, m$ and $C_j = 0$ for $j = 1, \ldots, n$. Since A is a non-zero matrix, there are indices i_0 and j_0 such that $a_{i_0,j_0} \neq 0$, but $R_{i_0} = 0$ and $C_{j_0} = 0$, which leads to a contradiction with the assumption that the statement of the problem is false. □

† If $I^- \neq \emptyset$, then $\sum_{(i,j)\in I^- \times J^+} a_{i,j} \leq \sum_{i \in I^-} R_i < 0$ — a contradiction. We can argue similarly to show $J^+ = \emptyset$.

Problem j13-I-3/j13-I-9. *Find the limit*

$$\lim_{n\to\infty} \sqrt{1+2\sqrt{1+3\sqrt{\cdots+(n-1)\sqrt{1+n}}}}\ .$$

(Moubinool Omarjee, Paris)

Origin. This problem comes from the list which Mr. Moubinool Omarjee from Paris sent to us in December 2002. (List 1. Problem 1.) ∼

Solution. Let

$$u_{m,n} = \sqrt{1+m\sqrt{1+(m+1)\sqrt{\cdots+(n-1)\sqrt{1+n}}}}\ .$$

We have

$$u_{m,n}^2 = 1 + m u_{m+1,n}\,,$$
$$u_{m,n}^2 - (m+1)^2 = m\big(u_{m+1,n} - (m+2)\big)\,.$$

Using the equality $|a-b| = |a^2-b^2|/|a+b|$ and inequality $u_{m,n}+m+1 \geq m+2$, we get

$$\big|u_{m,n} - m - 1\big| \leq \frac{m}{m+2}\big|u_{m+1,n} - (m+2)\big|\,.$$

We deduce that

$$|u_{2,n} - 3| \leq \frac{2}{4}\cdot\frac{3}{5}\cdot\ldots\cdot\frac{n-1}{n+1}\cdot|u_{n-1,n} - n|\,,$$
$$|u_{2,n} - 3| \leq \frac{6}{n(n+1)}\left(\sqrt{1+(n-1)\sqrt{1+n}} - n\right) = O\left(\frac{1}{n}\right).$$

So we get

$$\lim_{n\to\infty} u_{2,n} = 3\,.$$

□

Problem j13-I-4/j13-I-12. *Let A and B be complex Hermitian 2×2 matrices with pairs of eigenvalues (α_1, α_2) and (β_1, β_2), respectively. Determine all possible pairs (γ_1, γ_2) of eigenvalues of the matrix $C = A + B$.* (Charles University in Prague)

Solution. Recall that all eigenvalues of a Hermitian matrix are real numbers and that there exists an orthonormal basis consisting of eigenvectors of the matrix. As we can add a suffitiently large multiple of the identity matrix to both matrices A and B, we can suppose wlog that $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and that $\gamma_1, \gamma_2 > 0$ as well.

Let us also wlog suppose $\alpha_1 \geq \alpha_2,\ \beta_1 \geq \beta_2,\ \gamma_1 \geq \gamma_2$ and $\alpha_1 - \alpha_2 \geq \beta_1 - \beta_2$. By easy arguments, we can see

$$\gamma_1 + \gamma_2 = \operatorname{Tr} C = \operatorname{Tr} A + \operatorname{Tr} B = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 .$$

Further, it holds that

$$\gamma_1 \leq \alpha_1 + \beta_1 , \qquad \gamma_2 \geq \alpha_2 + \beta_2 .$$

(The first inequality can be seen if we rewrite it slightly: $\gamma_1 = \|C\| \leq \|A\| + \|B\| = \alpha_1 + \beta_1$. The second inequality follows if we consider the equality above and the first inequality together. — Alternatively, $\gamma_1 = \max(Cx, x)/(x, x) \leq \max(Ax, x)/(x, x) + \max(Bx, x)/(x, x) = \alpha_1 + \beta_1$ and $\gamma_2 = \min(Cx, x)/(x, x) \geq \min(Ax, x)/(x, x) + \min(Bx, x)/(x, x) = \alpha_2 + \beta_2$.) Later we will also prove the inequalities

$$\gamma_1 \geq \alpha_1 + \beta_2 , \qquad \gamma_2 \leq \beta_1 + \alpha_2$$

(in fact, it suffices to prove only the first one because the second one follows if we use the equality given above).

From these inequalities, we can see that $\gamma_1 \in [\alpha_1 + \beta_2, \alpha_1 + \beta_1]$. (The value of γ_2 has to be "complementary" to obtain the right value of the sum $\gamma_1 + \gamma_2$. It also worths noting that even if $\gamma_1 = \alpha_1 + \beta_2$, then still $\gamma_1 \geq \gamma_2 = \beta_1 + \alpha_2$. This follows from the assumption $\alpha_1 - \alpha_2 \geq \beta_1 - \beta_2$.) We will show that γ_1 can assume any value from the given interval $[\alpha_1 + \beta_2, \alpha_1 + \beta_1]$. Consequently, the set of all possible pairs (γ_1, γ_2) of eigenvalues of the matrix $C = A + B$ is

$$\big\{ (\gamma_1, \gamma_2) : \alpha_1 + \beta_2 \leq \gamma_1 \leq \alpha_1 + \beta_1,\ \gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \big\} .$$

To see this, let us put

$$A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \qquad B = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, \qquad P(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

The matrix A obviously has eigenvalues (α_1, α_2). The matrix $B(t) = P^{-1}(t) B P(t)$ obviously has eigenvalues (β_1, β_2). If we note that $P^{-1}(t) = P^{\mathrm{T}}(t)$ and define the matrix $C(t) = A + B(t)$, we have

$$C(0) = A + B = \begin{pmatrix} \alpha_1 + \beta_1 & 0 \\ 0 & \alpha_2 + \beta_2 \end{pmatrix}, \qquad C(\tfrac{\pi}{2}) = \begin{pmatrix} \alpha_1 + \beta_2 & 0 \\ 0 & \alpha_2 + \beta_1 \end{pmatrix}.$$

The matrix $C(0)$ has the eigenvalue $\gamma_1(0) = \alpha_1 + \beta_1$. (Note that $\gamma_1(0) \geq \gamma_2(0) = \alpha_2 + \beta_2$.) The matrix $C(\pi/2)$ has the eigenvalue $\gamma_1(\pi/2) = \alpha_1 + \beta_2$. (Note that $\gamma_1(\pi/2) \geq \gamma_2(\pi/2) = \alpha_2 + \beta_1$.) As both eigenvalues (γ_1, γ_2) of a matrix C depend continuously on the coefficients of the matrix, we deduce that $\gamma_1(t)$ is a continuous function. Consequently, it assumes every value from the interval $[\alpha_1 + \beta_2, \alpha_1 + \beta_1]$, which we wanted to demonstrate.

Now it only remains to prove the inequality $\gamma_1 \geq \alpha_1 + \beta_2$ for any two complex Hermitian matrices A and B. Let us recall that we still wlog suppose $\alpha_1 \geq \alpha_2 > 0,\ \beta_1 \geq \beta_2 > 0$ and $\gamma_1 \geq \gamma_2 > 0$. Let v_1 and v_2 denote the eigenvectors of the matrix A corresponding to the eigenvalues α_1 and α_2, respectively, and let w_1 and w_2 denote the eigenvectors of B corresponding to the eigenvalues β_1 and β_2, respectively. We can suppose that the bases $\{v_1, v_2\}$ and $\{w_1, w_2\}$ are orthonormal. So there exists some unitary matrix $U = \left(\begin{smallmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{smallmatrix}\right)$ such that

$$\begin{aligned} v_1 &= u_{11} w_1 + u_{12} w_2 , \\ v_2 &= u_{21} w_1 + u_{22} w_2 , \end{aligned} \qquad \text{and} \qquad \begin{aligned} w_1 &= \overline{u}_{11} v_1 + \overline{u}_{21} v_2 , \\ w_2 &= \overline{u}_{12} v_1 + \overline{u}_{22} v_2 . \end{aligned}$$

We will estimate γ_1 in the following way. First,

$$\gamma_1 = \sup\{\, \|Cx\| : \|x\| = 1 \,\} \geq \|Cv_1\|$$

where $\|\cdot\|$ denotes the Euclidean norm. (Let us justify the formula. Recall that $\gamma_1 = \max_{\|x\|=1}(Cx, x)$. Obviously, γ_1^2 is the greater eigenvalue of C^2. Consequently, it follows that $\gamma_1^2 = \max_{\|x\|=1}(C^2x, x)$. As C is Hermitian, we have $(C^2x, x) = x^*CCx = x^*C^*Cx = (Cx, Cx) = \|Cx\|^2$.) Second,

$$\begin{aligned} Cv_1 = (A+B)v_1 = \alpha_1 v_1 + \beta_1 u_{11} w_1 + \beta_2 u_{12} w_2 = (\alpha_1 + \beta_2)v_1 + (\beta_1 - \beta_2)u_{11} w_1 = \\ = \big(\alpha_1 + \beta_2 + (\beta_1 - \beta_2)u_{11}\overline{u}_{11}\big)v_1 + (\beta_1 - \beta_2)u_{11}\overline{u}_{21} v_2\,. \end{aligned}$$

As the vectors v_1 and v_2 are orthonormal and $(\beta_1 - \beta_2)u_{11}\overline{u}_{11} \geq 0$, we conclude

$$\begin{aligned} \gamma_1 \geq \|Cv_1\| = \sqrt{\big|\alpha_1 + \beta_2 + (\beta_1 - \beta_2)u_{11}\overline{u}_{11}\big|^2 + \big|(\beta_1 - \beta_2)u_{11}\overline{u}_{21}\big|^2} \geq \\ \geq \sqrt{\big|\alpha_1 + \beta_2 + (\beta_1 - \beta_2)u_{11}\overline{u}_{11}\big|^2} \geq \alpha_1 + \beta_2\,. \end{aligned}$$

□

Problem j13-II-1/j13-II-51. *Two square matrices A and B with real entries satisfy the conditions $A^{2002} = B^{2003} = E$ and $AB = BA$. Prove that $A + B + E$ is invertible. (The symbol E denotes the identity matrix.)* (University of Belgrade)

Solution. Let $(A + B + E)x = 0$ for some vector x, i.e., $(B + E)x = -Ax$. Then we have $-A^2x = A(B + E)x = (B + E)Ax = -(B + E)^2x$, and, continuing in this way, $(B + E)^kx = (-1)^kA^kx$. As $A^{2002} = E$, we get $(B + E)^{2002}x = x$, i.e.,

$$\big((B + E)^{2002} - E\big)x = \big(B^{2003} - E\big)x = 0\,.$$

(Recall $B^{2003} = E$.) In other words, taking that $p(t) = (t + 1)^{2002} - 1$ and $q(t) = t^{2003} - 1$ are polynomials, we have just got

$$p(B)x = q(B)x = 0\,.$$

But, since 2003 is a prime, $q(t)/(t - 1)$ is a primitive polynomial for all its roots, and therefore none of them is a root of the another monic polynomial $p(t)$ of degree 2002; further, the remained root $t = 1$ of $q(t)$ is not a root of $p(t)$, which implies that $p(t)$ and $q(t)$ are coprime.†

Since there exist non-zero polynomials $r(t)$ and $s(t)$ such that $r(t)p(t) - s(t)q(t) = 1$ (recall the Euclidean algorithm), we can conclude that $x = r(B)p(B)x - s(B)q(B)x = 0$, and so $A + B + E$ must be invertible indeed. □

† The polynomials $p(t)$ and $q(t)$ are really coprime (relatively prime). Here is another argument: Every polynomial (of degree ≥ 1) can be written as a product of factors of degree 1. In particular, $p(t) = (t + 1)^{2002} - 1 = \prod_{k=1}^{2002}(t - z_{p,k})$ and $q(t) = t^{2003} - 1 = \prod_{k=1}^{2003}(t - z_{q,k})$, where $z_{p,1}, \ldots, z_{p,2002}$ and $z_{q,1}, \ldots, z_{q,2003}$ are the roots of the polynomial p and q, respectively. Obviously, the polynomials p and q are relatively prime iff they have no root in common.

It is easy to see that the roots of q lie on the unit circle in the complex plane. Similarly, it is easy to see that all roots of p are on the circle with radius 1 and its centre at the point -1.

Thus, the intersections of the two circles,

$$-\tfrac{\sqrt{2}}{2} \pm \mathrm{i}\tfrac{\sqrt{2}}{2} = \cos \pm\tfrac{3\pi}{2} + \mathrm{i}\sin \pm\tfrac{3\pi}{2} = (-1) + \big(\cos \pm\tfrac{\pi}{2} + \mathrm{i}\sin \pm\tfrac{\pi}{2}\big)\,,$$

are the only possible common roots of q and p. But none of these two points is a root of q. It follows that p and q are coprime.

Problem j13-II-2/j13-I-17. *Let $\{D_1, D_2, \ldots, D_n\}$ be a set of disks in the Euclidean plane and $a_{ij} = S(D_i \cap D_j)$ be the area of $D_i \cap D_j$. Prove that for any numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$, the following inequality holds:*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j \geq 0\,.$$

(Warsaw University)

Solution. Let $\chi_{D_i}\colon \mathbb{R}^2 \to \{0,1\}$ be the characteristic function of the set D_i:

$$\chi_{D_i}(x,y) = \begin{cases} 1\,, & \text{if } (x,y) \in D_i\,, \\ 0\,, & \text{if } (x,y) \notin D_i\,. \end{cases}$$

We have

$$\chi_{D_i \cap Dj} = \chi_{D_i}\chi_{D_j}\,,$$

hence

$$S(D_i \cap D_j) = \iint_{\mathbb{R}^2} \chi_{D_i \cap D_j}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathbb{R}^2} \chi_{D_i}(x,y)\chi_{D_j}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Therefore

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j = \iint_{\mathbb{R}^2} \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\chi_{D_i}(x,y)x_j\chi_{D_j}(x,y)\,\mathrm{d}x\,\mathrm{d}y =$$
$$= \iint_{\mathbb{R}^2} \big(x_1\chi_{D_1}(x,y) + \cdots + x_n\chi_{D_n}(x,y)\big)^2\,\mathrm{d}x\,\mathrm{d}y \geq 0\,.$$

□

Problem j13-II-3/j13-II-70. *A sequence $(a_n)_{n=0}^{\infty}$ of real numbers is defined recursively by*

$$a_0 := 0\,, \qquad a_1 := 1\,, \qquad a_{n+2} := a_{n+1} + \frac{a_n}{2^n}\,, \quad n \geq 0\,.$$

Prove the following:

(a) *The sequence $(a_n)_{n=0}^{\infty}$ is convergent.*

(b)

$$\lim_{n\to\infty} a_n = 1 + \sum_{n=1}^{\infty} \frac{1}{2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n}(2^k - 1)}\,.$$

(c) *The limit $\lim_{n\to\infty} a_n$ is an irrational number.*

(Assistant (and former student competitor) Vjekoslav Kovač / University of Zagreb)

Solution. (a) Obviously, $a_n \geq 0$ for every $n \geq 0$. The sequence $(a_n)_{n=0}^{\infty}$ is increasing since $a_{n+2} - a_{n+1} = a_n/2^n \geq 0$ for every $n \geq 0$. It suffices to show that $(a_n)_{n=0}^{\infty}$ is bounded from above. For each $n \geq 0$, we have $a_{n+2} \leq a_{n+1} + a_{n+1}/2^n = a_{n+1}(1 + 1/2^n)$. Using the inequality between geometric and arithmetic mean, for every $n \geq 1$ we obtain

$$a_{n+2} \leq \prod_{k=0}^{n}\Big(1 + \frac{1}{2^k}\Big) = 2\prod_{k=1}^{n}\Big(1 + \frac{1}{2^k}\Big) \leq 2\bigg(\frac{1}{n}\Big(n + \sum_{k=1}^{n}\frac{1}{2^k}\Big)\bigg)^n \leq 2\Big(\frac{n+1}{n}\Big)^n \leq 2\mathrm{e}\,.$$

(b) Consider the power series $\sum_{n=0}^{\infty} a_n z^n$. Since $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq \lim_{n\to\infty} \sqrt[n]{2\mathrm{e}} = 1$, its radius of convergence is $R \geq 1$. Therefore, on the open unit disc, with center at the origin, it converges to a holomorphic function $f(z) := \sum_{n=0}^{\infty} a_n z^n$. Inductively, we obtain $a_{n+2} = 1 + \sum_{k=0}^{n} a_k/2^k$ for any $n \geq 0$. So $\lim_{n\to\infty} a_n = 1 + \sum_{k=0}^{\infty} a_k/2^k = 1 + f\big(\frac{1}{2}\big)$ and we have to find $f\big(\frac{1}{2}\big)$.

Now we use the recurrent relation for $(a_n)_{n=0}^{\infty}$ to obtain a functional equation for f. We multiply $a_{n+2} := a_{n+1} + a_n/2^n$ by z^{n+2} and sum over all $n \geq 0$ to get

$$\sum_{n=0}^{\infty} a_{n+2} z^{n+2} = z\sum_{n=0}^{\infty} a_{n+1} z^{n+1} + z^2 \sum_{n=0}^{\infty} a_n \big(\tfrac{z}{2}\big)^n,$$

that is

$$f(z) - z = zf(z) + z^2 f\big(\tfrac{z}{2}\big)\,,$$

or

$$(1 - z)f(z) = z^2 f\big(\tfrac{z}{2}\big) + z \qquad \text{for } |z| < 1\,. \tag{1}$$

We substitute $z = 1/2^n$ for $n = 1, \ldots, N$ (where $N \geq 1$ is a fixed number) into (1), then multiply the n-th equality by some constant $s_n > 0$ and finally sum up those N equalities:

$$\begin{array}{lclr}
(1 - \frac{1}{2})\, f\big(\frac{1}{2}\big) &=& \big(\frac{1}{2}\big)^2 f\big(\frac{1}{4}\big) + \frac{1}{2}\,, & |\cdot s_1\,, \\
(1 - \frac{1}{4})\, f\big(\frac{1}{4}\big) &=& \big(\frac{1}{4}\big)^2 f\big(\frac{1}{8}\big) + \frac{1}{4}\,, & |\cdot s_2\,, \\
&\vdots&& \\
(1 - \frac{1}{2^n})\, f\big(\frac{1}{2^n}\big) &=& \big(\frac{1}{2^n}\big)^2 f\big(\frac{1}{2^{n+1}}\big) + \frac{1}{2^n}\,, & |\cdot s_n\,, \\
(1 - \frac{1}{2^{n+1}}) f\big(\frac{1}{2^{n+1}}\big) &=& \big(\frac{1}{2^{n+1}}\big)^2 f\big(\frac{1}{2^{n+2}}\big) + \frac{1}{2^{n+1}}\,, & |\cdot s_{n+1}\,, \\
&\vdots&& \\
(1 - \frac{1}{2^N})\, f\big(\frac{1}{2^N}\big) &=& \big(\frac{1}{2^N}\big)^2 f\big(\frac{1}{2^{N+1}}\big) + \frac{1}{2^N}\,, & |\cdot s_N\,, \\
\hline
\frac{s_1}{2}\, f\big(\frac{1}{2}\big) &=& \frac{s_N}{2^{2N}}\, f\big(\frac{1}{2^{N+1}}\big) + \sum_{n=1}^{N} \frac{s_n}{2^n}\,. &
\end{array}$$

To obtain the given result (namely, to achieve cancelling of the terms with $f\big(\frac{1}{2^n}\big)$ for $n = 2, \ldots, N$), we had to choose the numbers s_n so that

$$\big(1 - \tfrac{1}{2^{n+1}}\big)s_{n+1} = \big(\tfrac{1}{2^n}\big)^2 s_n \qquad \text{for } n \geq 0\,. \tag{2a}$$

Let us put

$$s_0 := 1\,. \tag{2b}$$

It follows that $s_1 = 2$. Equalities (2b) and (2a) lead to

$$s_n = \prod_{k=0}^{n-1} \frac{s_{k+1}}{s_k} = \prod_{k=0}^{n-1} \frac{\left(\frac{1}{2^k}\right)^2}{1-\frac{1}{2^{k+1}}} = \prod_{k=0}^{n-1} \frac{1}{2^{k-1}(2^{k+1}-1)} = \frac{1}{2^{\frac{n(n-1)}{2}-n}\prod_{k=1}^{n}(2^k-1)}$$

for every $n \geq 1$. Finally, we have

$$f\left(\tfrac{1}{2}\right) = \frac{s_N}{2^{2N}} f\left(\tfrac{1}{2^{N+1}}\right) + \sum_{n=1}^{N} \frac{s_n}{2^n} = \frac{f\left(\frac{1}{2^{N+1}}\right)}{2^{\frac{N(N-1)}{2}+N}\prod_{k=1}^{N}(2^k-1)} + \sum_{n=1}^{N} \frac{1}{2^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}(2^k-1)}\,.$$

The first term tends to 0 when $N \to \infty$, so

$$f\left(\tfrac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{2^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}(2^k-1)}\,. \tag{3}$$

(c) The proof of $\lim_{n\to\infty} a_n \in \mathbb{R}\setminus\mathbb{Q}$ is based on the fact that the series in (3) converges "very rapidly". Suppose that its sum equals $\frac{p}{q}$ for some positive integers p and q. For each integer $N \geq 1$, denote

$$q_N := 2^{\frac{N(N-1)}{2}} \prod_{k=1}^{N}(2^k-1)\,, \qquad p_N := q_N \sum_{n=1}^{N} \frac{1}{2^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}(2^k-1)}\,.$$

Obviously, p_N and q_N are positive integers. We manage to estimate $pq_N - qp_N$. We have

$$q_N = 2^{\frac{N(N-1)}{2}} \prod_{k=1}^{N}(2^k-1) < 2^{\frac{N(N-1)}{2}} \prod_{k=1}^{N} 2^k = 2^{N^2}$$

and

$$\frac{p}{q} - \frac{p_N}{q_N} = \sum_{n=N+1}^{\infty} \frac{1}{2^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}(2^k-1)} \leq \sum_{n=N+1}^{\infty} \frac{1}{2^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}2^{k-1}} =$$
$$= \sum_{n=N+1}^{\infty} \frac{1}{2^{n(n-1)}} \leq \sum_{m=N(N+1)}^{\infty} \frac{1}{2^m} = \frac{1}{2^{N^2+N-1}} < \frac{1}{2^{N-1}q_N}\,.$$

Thus, $0 < pq_N - qp_N < \frac{q}{2^{N-1}}$,† so $(pq_N - qp_N)_{N\geq 1}$ is a sequence of positive integers that converges to 0. This is a contradiction and we are done. □

† It is easy to see from the definition of the numbers p_N that the sequence $\left(\frac{p_N}{q_N}\right)$ is strictly increasing to the limit $\frac{p}{q}$. Hence $\frac{p_N}{q_N} < \frac{p}{q}$, $qp_n < pq_N$, and $0 < pq_N - qp_N$. As the difference is integer, we have even $1 \leq pq_N - qp_N$.

Problem j13-II-4/j13-I-18. *Let $f, g\colon [0,1] \to (0,+\infty)$ be continuous functions such that f and $\frac{g}{f}$ are increasing. Prove that*

$$\int_0^1 \frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t}\,\mathrm{d}x \le 2\int_0^1 \frac{f(t)}{g(t)}\,\mathrm{d}t\,.$$

(Assistant (and former student competitor) Vjekoslav Kovač / University of Zagreb)

Solution. First, we estimate the expression inside the integral sign on the left side of the given inequality. By the Chebycheff's inequality for integrals applied to increasing functions f and $\frac{g}{f}$ on the segment $[0,x]$ (where $x \in (0,1]$ is fixed), we get

$$\Big(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\Big)\Big(\frac{1}{x}\int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t\Big) \le \frac{1}{x}\int_0^x g(t)\,\mathrm{d}t\,,$$

that is,

$$\frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t} \le \frac{x}{\int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t} \tag{1}$$

for every $x \in (0,1]$. From the integral form of the Cauchy-Schwarz inequality on the segment $[0,x]$, we have

$$\Big(\int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t\Big)\Big(\int_0^x \frac{t^2 f(t)}{g(t)}\,\mathrm{d}t\Big) \ge \Big(\int_0^x t\,\mathrm{d}t\Big)^2 = \frac{x^4}{4}\,,$$

or

$$\frac{1}{\int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t} \le \frac{4}{x^4}\int_0^x \frac{t^2 f(t)}{g(t)}\,\mathrm{d}t\,. \tag{2}$$

From (1) and (2) we obtain

$$\frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t} \le \frac{4}{x^3}\int_0^x \frac{t^2 f(t)}{g(t)}\,\mathrm{d}t\,. \tag{3}$$

Finally, it remains to integrate (3) over $x \in (0,1]$ and to reverse the order of integration.

$$\begin{aligned}\int_0^1 \frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t}\,\mathrm{d}x &\le \int_0^1 \Big(\int_0^x \frac{4t^2 f(t)}{x^3 g(t)}\,\mathrm{d}t\Big)\,\mathrm{d}x = \int_0^1 \Big(\int_t^1 \frac{4t^2 f(t)}{x^3 g(t)}\,\mathrm{d}x\Big)\,\mathrm{d}t = \\ &= \int_0^1 \frac{4t^2 f(t)}{g(t)}\Big(\int_t^1 \frac{\mathrm{d}x}{x^3}\Big)\,\mathrm{d}t = \int_0^1 \frac{4t^2 f(t)}{g(t)}\Big(\frac{1}{2t^2} - \frac{1}{2}\Big)\,\mathrm{d}t = \\ &= 2\int_0^1 \frac{f(t)}{g(t)}(1-t^2)\,\mathrm{d}t \le 2\int_0^1 \frac{f(t)}{g(t)}\,\mathrm{d}t\,.\end{aligned}$$

□

Remark. The constant 2 on the right hand side of the given inequality is optimal, i.e., the least possible one. Consider $f(t) := 1$ and $g(t) := t + \varepsilon$ for some fixed $\varepsilon > 0$. Then

$$\int_0^1 \frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t}\,\mathrm{d}x = \int_0^1 \frac{x}{\frac{1}{2}x^2 + \varepsilon x}\,\mathrm{d}x = 2\int_0^1 \frac{\mathrm{d}x}{x + 2\varepsilon} = 2\ln(1+2\varepsilon) - 2\ln 2 - 2\ln\varepsilon$$

and

$$\int_0^1 \frac{f(t)}{g(t)}\,\mathrm{d}t = \int_0^1 \frac{\mathrm{d}t}{t+\varepsilon} = \ln(1+\varepsilon) - \ln\varepsilon.$$

The quotient of these two expressions can be made arbitrarily close to 2 since

$$\lim_{\varepsilon \searrow 0} \frac{2\ln(1+2\varepsilon) - 2\ln 2 - 2\ln\varepsilon}{\ln(1+\varepsilon) - \ln\varepsilon} = 2\lim_{\varepsilon \searrow 0} \frac{-\frac{\ln(1+2\varepsilon)}{\ln\varepsilon} + \frac{\ln 2}{\ln\varepsilon} + 1}{-\frac{\ln(1+\varepsilon)}{\ln\varepsilon} + 1} = 2\,.$$

Therefore, the constant 2 is the best possible one. ∼